International Journal of Theoretical Physics, Vol. 25, No. 3, 1986

Toward a Comprehensive Theory for He II. II. A Temperature-Dependent Field-Theoretic Approach

J. Chela-Flores[1,2] and H. B. Ghassib[3]

Received June 12, 1984

New experimental aspects of He II are used as a guide toward a comprehensive theory in which nonzero temperature $U(1)$ and $SU(2)$ gauge fields are incorporated into a gauge hierarchy of effective Lagrangians. We conjecture that an $SU(n)$ gauge-theoretic description of the superfluidity of ^4He may be obtained in the limit $n \to \infty$. We indicate, however, how experiments may be understood in the zeroth-, first-, and second-order of the hierarchy.

1. INTRODUCTION

1.1. The Variational Principle in Low-Temperature Physics

The necessity for appealing to a powerful field-theoretic approach in the problems of low-temperature physics was recognized a long time ago. Cook (Cook, 1940) used Eckart's variational principle (Eckart, 1938) in order to obtain the London equations of superconductivity (London and London, 1935). More recently, the full microscopic theory (Bardeen et al., 1957) has been studied in a Lagrangian formalism (Nambu, 1960; Lurie, 1970; Chela-Flores, 1974; Baldo et al., 1977). The hydrodynamic equations of the two-fluid model developed by Tisza for He II were inferred from the Hamilton principle of particle mechanics (Tisza, 1947), but some difficulties were pointed out, including the failure by the theory to satisfy the law of conservation of momentum (Landau, 1949). Besides, a significant remark was made by Zilsel (Zilsel, 1950), namely, that Tisza's equations are valid only in the limit of low velocities. Zilsel went on to derive the dissipationless two-fluid equations of motion for liquid helium-four, giving explicitly its

[1]Departamento de Física, Universidad Simón Bolívar, Apartado 80659, Caracas, Venezuela.
[2]Also at Instituto Internacional de Estudios Avanzados (I.D.E.A.), Caracas, Venezeula.
[3]Department of Physics, University of Jordan, Amman, Jordan.

0020-7748/86/0300-0273$05.00/0

thermodynamic and hydrodynamic properties and incorporating the interactions of the elementary excitations with a very wide range of applicability, particularly with regard to velocity and temperature. Jackson (1978, 1979) has discussed some criticisms put forward against Zilsel's approach (Lim, 1963; Lhuillier et al., 1975) and has given a new Lagrangian derivation.

1.2. Recent Developments in High-Energy Physics

The above applications of the action-principle formalism benefit from a direct connection between symmetry principles and conservation laws. It is therefore very convenient to formulate physical laws by means of such variational methods.

During the last few years considerable progress in understanding field theory has taken place, mainly with respect to the problem of the gravitational force (Fayet, 1982) as well as that of high-energy collisions. The latter has led to a theory of quark interactions–quantum chromodynamics (Gross and Wilczek, 1973; Politzer, 1973). It is therefore natural to enquire whether such progress in the physics of high energies can throw some new light on the remaining problems in the physics of low temperatures.

These recent field-theoretic developments go beyond the straightforward Lagrangian formulations mentioned above in the context of superconductivity and superfluidity. New concepts, rich in content, have invaded realistic field theories, such as non-Abelian gauge fields (Yang and Mills, 1954) and various applications of topology (Lee, 1981). While the latter concept has entered the domain of low-temperature physics with respect to the problem of superfluidity of ^{3}He (Golo and Monastyrski, 1978), there are yet no applications of the concepts introduced by Yang and Mills (Yang and Mills, 1954).

For the benefit of the uninitiated reader from low-temperature physics, and in view of the further developments we expect in future work, we have seen it appropriate to include the rather technical appendix (although it reproduces well-known work).

1.3. Recent Experimental Developments in He II

Recent developments have also occurred in our experimental understanding of He II, in particular the following:

(i) Unusually high velocities should be incorporated in a hydrodynamic description of Landau's two-fluid model, since critical velocities for superfluid flow through micron-size orifices were observed by Schofield (1972) to reach values as high as 10 m sec^{-1}.

(ii) Propagation of collective waves on the surface of He II films, one atomic layer thick, has been observed (Scholtz et al., 1974).

(iii) A better understanding of the He II atomic order has been accumulating (Robkoff and Hallock, 1981), so that unprecedented accuracy at large wave-vector transfer Q and low-temperature T is now available. In addition, measurements for $S(Q, T)$ in the temperature range $1.38 < T < 4.24$ K at s.v.p. and $Q = 5.1$ Å^{-1} have been undertaken.

(iv) Neutron scattering determination of the momentum distribution function $n(\mathbf{p})$ at large Q has presented us with a very strong experimental result which a theory of He II must describe. Sears et al. (1982) report that at 1 K about 13% of all atoms are in their lowest mode.

1.4. Theoretical Implications of Recent Experiments

New constraints arise in the theory in view of the recent results (i)–(iv). From (i) it is clear that proper account must be taken in a Zilsel-like approach of interactions among elementary excitations in such a way that high-fluid velocities may come within the scope of theory. More precisely, transitions between the two fluids are important (Jackson, 1979), thus going beyond Landau's theory (Landau, 1941; Landau, 1944).

We may interpret (ii) as evidence in favor of taking the hydrodynamics of He II as having a greater range of validity than the Navier–Stokes theory for classical fluids (Putterman, 1974). A mixing of scales is to be expected: quantum effects occur on a macroscopic level while ^{4}He hydrodynamics *extends its range of validity* to a microscopic level. It follows that Landau's model does not provide the required quantum continuum theory (Putterman, 1974; Chela-Flores, 1976).

The new experiments (iii) and (iv) are closely related. They underline the fact that, regardless of whether London's point of view (London, 1938a; London, 1938) on the microscopic origin of superfluidity is correct or not, the eventual microscopic theory we are still searching for must address itself to the "counting process," i.e., to the determination of the momentum distribution function $n(\mathbf{p})$ for the lowest $\mathbf{p}$, as well as to the shape of $S(Q, T)$ for all Q and all T ($\leq T_\lambda$). The accuracy of the recent experiments (Robkoff and Hallock, 1981; Sears et al., 1982) lead us to have such expectations of the predictive power of the eventual microscopic theory.

1.5. Toward a Complete Description of He II

Our point of view in the present work, as well as in that immediately preceding it (Ghassib and Khudeir, 1986), is closely related to Jackson's approach to liquid ^{4}He: we feel that a unified theory of superfluidity is necessary in which previous successful approaches are incorporated into the new work. Thus all the good features of the preceding theory adopted will become good features of the new comprehensive theory. However, the new work will, in some manner, extend its range of validity so as to address itself to the new phenomena being explored in the 1980s.

Jackson has based his new work on the well-known description of He II in terms of correlated basis functions, which accounts for experimentally determined properties of liquid ^{4}He with some success (Jackson and Feenberg, 1961; Jackson and Feenberg, 1962; Feenberg, 1969).

1.6. The Hartree Liquid as a Limiting Case of an Abelian Gauge Theory

On the other hand, the simple Hartree liquid approach to He II has been equally successful in incorporating a large number of features of He II (Gross, 1966; Pitaevskii, 1961). By its very formulation, it lends itself easily to treatment with the conventional methods of field theory, as it has been impressively shown recently by Anandan (Anandan, 1981, 1984), where he raises the possibility of using He II as the means for the first laboratory test of general relativity. We hope to show in Section 2 below that the Hartree liquid model (Gross, 1966; Pitaevskii, 1961) may serve as the limiting case of a temperature-dependent Abelian gauge theory of He II as the corresponding gauge field decouples. In this manner we set up a program of work in which we are guaranteed that the new more comprehensive theory to be constructed will carry over the proper description of ion mobility in the bulk fluid, vortex structure, critical-velocity phenomena, and so forth (Gross, 1966).

Having in mind the recent experimental results (i)–(iv) of Section 1.3, we prefer, following Gross (1966), to begin with the wave function for the N-helium atom system of real He II, $\psi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N; t)$. We then use the Hartree approximation, according to which *each* single-particle wave function ψ must satisfy

$$i\partial_t\psi = \frac{\hbar^2}{2m_4^*}\partial_{\mathbf{x}}^2\psi(\mathbf{x}, t) + \frac{\tilde{\lambda}}{2}\psi(\mathbf{x}, t) + \psi(\mathbf{x}, t)\int d\mathbf{x}'\, V(\mathbf{x}-\mathbf{x}')|\psi(\mathbf{x}, t)|^2 \quad (1)$$

Here $\tilde{\lambda}$ denotes an average energy, while V denotes the effective interaction which each single particle is subject to by the effect of the others; finally, m_4^* denotes the mass of a ^{4}He quasiparticle.

In view of what was said in Section 1.3, we shall attempt to construct a Lagrangian density which reproduces (1), but which will lead us to a temperature-dependent gauge theory by turning gauge invariance of the first kind into that of the second kind, as explained by Utiyama (1956). In the case of zero temperature, this has already been done (Chela-Flores, 1975):

$$\mathscr{L}_{\text{eff}}^{(0)} = -\frac{i}{2}(\psi\partial_t\psi^* - \psi^*\partial_t\psi) - \frac{1}{2}\partial_{\mathbf{x}}\psi^* \cdot \partial_{\mathbf{x}}\psi$$

$$-\int \psi^*(\mathbf{x}' \cdot t)\psi^*(\mathbf{x}, t)V(\mathbf{x}-\mathbf{x}')\psi(\mathbf{x}, t)\psi(\mathbf{x}', t)\,d\mathbf{x}' - \frac{1}{2}\tilde{\lambda}\psi^*\psi \qquad (2)$$

Here we have considered the Lagrangian density as the zeroth approximation to the hierarchy of effective Lagrangians which will describe He II. In the above equation, and henceforth, we use a system of units such that

$$h = m_4^* = k_B = 1$$

where k_B is Boltzmann's constant. In equation (2) we have introduced trivial corrections to the original form of the Lagrangian density, which were pointed out (Chela-Flores et al., 1977) after the publication of the $U(1)$ gauge theory (Chela-Flores, 1975).

In order to complete the program initiated at the beginning of Section 1.5, the rest of this paper is laid out as follows. In Section 2 we give the basic arguments required to generalize the earlier formalism (Chela-Flores, 1975) so as to include temperature variations. We follow this, in Section 3, with a brief account of the hydrodynamics, indicating the differences and similarities between the zero-T and $T \neq 0$ theory, as well as between the limiting case of the uncoupled gauge field and that in the presence of this field.

In Section 4 we outline the details of a calculation of the liquid structure factor at this *lowest* stage of the hierarchy of effective Lagrangians of the full comprehensive approach to He II. In Section 5 we sketch the second stage of the proposed hierarchy by developing a non-Abelian gauge theory with a limiting case (uncoupled field) coinciding with a pairing theory of bosons with an energy gap. In Section 6 we conjecture that the hierarchy of successive effective lagrangians for He II leads to a general $SU(n)$ gauge theory. Finally, in Section 7, we summarize and conclude.

2. A TEMPERATURE-DEPENDENT $U(n=1)$ GAUGE THEORY

The main idea is to regard our system of strongly interacting particles as ar ıbly of weakly interacting *quasiparticles* (Bloch, 1965). The conventional methods of statistical mechanics, formulated in terms of the grand canonical ensemble, can then be directly applied in the usual manner (Valatin and Butler, 1958; Pathria, 1972). Since many of the resulting equations are in a more or less one-to-one correspondence with their zero-temperature counterpart we shall be quite brief in presenting the formal background. The function $\psi(\mathbf{x}, t)$ denotes the expectation value $\langle\psi\rangle$ of the annihilation field operator, the averaging procedure being performed in the familiar statistical-mechanical manner for a gas of Bose quasiparticles within the framework of the grand canonical ensemble:

$$\langle\psi\rangle = T_r\{\exp[-\beta\hat{H}(T)]\hat{\psi}\}/Z_G \tag{3}$$

where Z_G is the grand partition function:

$$Z_G = T_r \exp[-\beta\hat{H}(T)] \tag{4}$$

and β is the usual temperature parameter, which is simply T^{-1} in the present system of units.

By turning the gauge invariance of the first kind (global symmetry) of the Lagrangian density (2) into gauge invariance of the second kind (local symmetry), we are led to a new action principle:

$$\delta \int \mathscr{L}_{\text{eff}}^{(1)}\, d_3x = 0 \tag{5}$$

where the effective Lagrangian density for the first stage of the gauge hierarchy is given by

$$\mathscr{L}_{\text{eff}}^{(1)} = -\frac{i}{2}(\psi\partial_t\psi^* - \psi^*\partial_t\psi) - \frac{\tilde{\lambda}}{2}\psi^*\psi - \frac{1}{2}[(\partial_{\mathbf{x}} + i\mathbf{u}) - \tilde{\lambda}]\psi^* \cdot [(\partial_{\mathbf{x}} - i\mathbf{x}) - \tilde{\lambda}]\psi$$
$$- \int \psi^*(\mathbf{x}', t)\psi^*(\mathbf{x}, t)\, V(\mathbf{x} - \mathbf{x}')\psi(\mathbf{x}, t)\psi(\mathbf{x}', t)\, dx' + \frac{k}{2}\partial_{\mathbf{x}} \times \mathbf{u} \cdot \partial_{\mathbf{x}} \times \mathbf{u} \tag{6}$$

As usual, the simplest scalar Lagrangian density of the gauge field $\mathbf{u}$,

$$\mathscr{L}_0[\mathbf{u}] \equiv \frac{k}{2}\partial_{\mathbf{x}} \times \mathbf{u} \cdot \partial_{\mathbf{x}} \times \mathbf{u} \tag{7}$$

has been added here, and derivatives have been replaced by their covariant counterparts. The constant $[m_4^*](k/2)$ is required for dimensional consistency.

We work in terms of a temperature-dependent Hamiltonian $H(T)$ adjusted to yield the chemical potential $\mu = \mu(T)$:

$$\hat{H}(T) = \hat{H} - \mu\hat{N} \tag{8}$$

$\hat{H}$ denoting the original Hamiltonian of the system, inferred from the above Lagrangian density through the standard variational method, and

$$\hat{N} = \sum_{\mathbf{k}} a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} \tag{9}$$

denoting the number operator.

In equation (2) the energy parameter $\tilde{\lambda}$ is given by

$$\tilde{\lambda} = \mu N + T\sigma \tag{10}$$

σ being the entropy of the weakly interacting gas of Bose *quasiparticles*; this is related to the Bose–Einstein distribution function $f_{\mathbf{k}}$ through the equation (Valatin and Butler, 1958):

$$\sigma = -\sum_{\mathbf{k}} [f_{\mathbf{k}} \ln f_{\mathbf{k}} - (1+f_{\mathbf{k}}) \ln(1+f_{\mathbf{k}})] \tag{11}$$

where

$$f_{\mathbf{k}} = [\exp \beta(\varepsilon_{\mathbf{k}} - \mu) - 1]^{-1} \tag{12}$$

$\varepsilon_{\mathbf{k}}$ being the quasiparticle energy.

The corresponding Euler–Lagrangian equations of motion are

$$\begin{aligned} i\partial_t \psi_T(\mathbf{x}, t) = &-\frac{1}{2}[(\partial_{\mathbf{x}} - i\mathbf{u})^2 - \tilde{\lambda}]\psi_T(\mathbf{x}, t) \\ &+ \int V(\mathbf{x}-\mathbf{x}')|\psi_T(\mathbf{x}', t)|^2 \psi_T(\mathbf{x}, t)\, d\mathbf{x}' \end{aligned} \tag{13}$$

and

$$\begin{aligned} k\partial_{\mathbf{x}} \times \partial_{\mathbf{x}} \times \mathbf{u} = &-\frac{1}{2i}[\psi_T^*(\mathbf{x}, t)\, \partial_{\mathbf{x}} \psi_T(\mathbf{x}, t) - \psi_T(\mathbf{x}, t)\, \partial_{\mathbf{x}} \psi_T^*(\mathbf{x}, t)] \\ &- |\psi_T(\mathbf{x}, t)|^2 \mathbf{u} \end{aligned} \tag{14}$$

where a subscript T has now been attached to ψ to denote its implicit dependence on temperature.

The underlying theory is a two-fluid picture (Chela-Flores, 1975); the field $\mathbf{u}$ may be conjectured to represent the velocity field of one of these two fluids. It would be fruitful, at some later stage, to clarify this picture by unmasking the explicit connection of ψ and $\mathbf{u}$ with thermodynamic quantities, in the manner of Landau's two-fluid theory (Landau, 1941, 1947). For the moment we have incorporated superfluidity into the system via the additional gauge symmetry which goes beyond that of the Hartree-liquid model. Accordingly, superfluidity and Bose–Einstein condensation are not necessarily interwoven in the present framework, whose generality may best be appreciated by viewing the Lagrangian ansatz of equation (6) as merely *a first step in a hierarchy of more and more refined effective Lagrangians*, which result as the system is explored further (Ghassib and Khudeir, 1986) (cf. Sections 5 and 6).

In the limit of vanishing gauge field **u**, equation (13) simply reduces to a temperature-dependent nonlinear Schrödinger equation:

$$i\partial_t\psi_T(\mathbf{x}, t) = -\frac{1}{2}(\partial_{\mathbf{x}}^2 - \tilde{\lambda})\psi_T(\mathbf{x}, t) + \psi_T(\mathbf{x}, t)\int V(\mathbf{x}-\mathbf{x}')|\psi_T(\mathbf{x}', t)|^2\, d\mathbf{x}' \tag{15}$$

Thus, in the limiting case, the $U(1)$ gauge theory combines the most attractive features of both the present $U(1)$ approach, as well as the weak coupling type of theory (cf. Section 1.5). In this sense, it is a temperature-dependent version of the so-called hybrid approach (Ghassib and Khudeir, 1986).

3. $U(n=1)$ HYDRODYNAMICS

The hydrodynamic equations follow from the familiar Madelung transformation (Gross, 1966):

$$\psi_T(\mathbf{x}, t) = R_T(\mathbf{x}, t)\exp[iS_T(\mathbf{x}, t)] \tag{16}$$

where $R_T(\mathbf{x}, t)$ and $S_T(\mathbf{x}, t)$ are both real quantities: R_T is the square root of the local fluid density. We now look at two different cases.

3.1. The Hartree-Liquid Limit

We shall first derive the hydrodynamic equations in the limiting case of a vanishingly small gauge field (low-velocity approximation). Substituting equation (16) into (15) and equating imaginary parts, we have

$$\partial_t R_T^2(\mathbf{x}, t) + \partial_{\mathbf{x}}\cdot[R_T(\mathbf{x}, t)\,\partial_{\mathbf{x}}S_T(\mathbf{x}, t)] = 0 \tag{17}$$

whereas from the real parts, we obtain

$$-R_T(\mathbf{x}, t)\,\partial_t S_T(\mathbf{x}, t) + \tfrac{1}{2}\partial_{\mathbf{x}}^2 R_T(\mathbf{x}, t) = E[V]R_T(\mathbf{x}, t) \tag{18}$$

where

$$E[V] = \{\tfrac{1}{2}[\partial_{\mathbf{x}}S_T(x, t)]^2 - \tilde{\lambda}\} + \int V(\mathbf{x}-\mathbf{x}')R_T^2(\mathbf{x}', t)\, d\mathbf{x}' \tag{19}$$

This is a generalized, temperature-dependent functional of the interaction.

We note that equation (17) is again the continuity equation, which is identical in form to its zero-temperature counterpart (Chela-Flores, 1975), save for the implicit temperature dependence of R_T and S_T. Further, equation (17) is the temperature-dependent Bernoulli equation, which is extremely rich in content.

3.2. The U(1) Full Equations

We shall retain the gauge field, as was done in the $U(1)$ zero-temperature case (Chela-Flores, 1975). Taking imaginary parts leads us once again to a Bernoulli equation (18), with a correcting term including the gauge field.

However, we wish to show explicitly the effect of taking real parts. In this case, we find

$$\partial_t \rho + \partial_{\mathbf{x}} \cdot (\rho \mathbf{v}) = \tfrac{1}{2}\mathbf{u} \cdot \partial_{\mathbf{x}} \rho \tag{20}$$

This leads us to expect a *two-fluid picture*, in which $\rho(\mathbf{x}, t, T)$ is the density of one fluid which, when added to the density of a second interpenetrating fluid of density $\tau(\mathbf{x}, t, T)$, retrieves the total bulk density ρ_{tot}:

$$\rho_{\text{tot}} = \rho(\mathbf{x}, t, T) + \tau(\mathbf{x}, t, T) \tag{21}$$

It follows that the hydrodynamic continuity equation for the second fluid is

$$\partial_t \tau + \partial_{\mathbf{x}} \cdot (\tau \mathbf{u}) = -\tfrac{1}{2}\mathbf{u} \cdot \partial_{\mathbf{x}} \tau \tag{22}$$

for, in this case, we recover the expected continuity equation for the bulk fluid:

$$\partial_t \rho_{\text{tot}} + \partial_{\mathbf{x}} \cdot \mathbf{J} = 0 \tag{23}$$

where the total current is given by

$$\mathbf{J} = \rho(\mathbf{x}, t, T)\mathbf{v} + \tau(\mathbf{x}, t, T)\mathbf{u} \tag{24}$$

Thus, just as in the $T = 0$ case, at elevated temperatures we are also led to the Fröhlich relation (Fröhlich, 1969):

$$\partial_t \tau + \partial_{\mathbf{x}} \cdot (\tau \mathbf{u}) = -[\partial_t \rho + \partial_{\mathbf{x}} \cdot (\rho \mathbf{v})] \neq 0 \tag{25}$$

Yet it should be stressed that our equations (20) and (22) do not coincide with the Fröhlich two-fluid theory for He II. Nor does our two-fluid picture duplicate Landau's theory (Landau, 1941) either. Yet, the possibility of implementing Putterman's program (1974) within our theory is not ruled out (see Sections 1 and 4). Since equations (16)-(24) are formally identical to the zero-temperature limit already studied (Chela-Flores, 1975), it follows that linear and circular vortices may be suitably described. We prefer to refer the reader to the published work and explore in the next section, as an illustration, some of the features of the liquid structure factor, then demonstrate how the present Abelian gauge theory should be viewed only as a first stage in a hierarchy of ever more accurate *effective* Lagrangians of possibly non-Abelian gauge theories. We shall return to this important remark in Section 6.

4. $U(n=1)$ APPROXIMATE CALCULATION OF ATOMIC ORDER

As a first step, and for comparison purposes, we adopt the same set of approximations already used in previous work at zero temperature (Chela-Flores, 1977; Ghassib and Khudeir, 1986), namely, the following:

(a) The term $\frac{1}{2}\partial_x S_T(\mathbf{x}, t)$ is neglected, implying the low-fluid-velocity limit.

(b) A stationary fluid is assumed, in the sense that $-\partial_t S_T(\mathbf{x}, t)$ is set equal to a constant, E_v, say.

(c) A spherically symmetric solution is used, that is, $\rho(x, T) = \rho(r, T) \equiv R_T^2(r)$.

(d) We confine our attention to the long-wavelength (low-Q) limit only.

(e) We employ, once again, a purely δ-function interaction:

$$V(\mathbf{x}-\mathbf{x}') = U\delta(\mathbf{x}-\mathbf{x}') \tag{26}$$

While approximations (a)-(d) are physically quite sound, (e) has been adopted only for mathematical convenience. It must therefore be abandoned eventually in favor of a more realistic He-He interaction if the proper contact with experiment is to be made.

With this set of approximations, equations (18) and (19) reduce to a nonlinear differential equation for $\rho(r, T)$:

$$2\rho^{3/2}U - 2\tilde{E}_v\rho^{1/2} - \partial_r(\rho^{1/2}) = 0 \tag{27}$$

where

$$\tilde{E}_v \equiv E_v + \tilde{\lambda} = E_v + \mu N + T_\sigma \tag{28}$$

Equation (27) can be solved by direct integration (Chela-Flores, 1977) to yield, after some trivial manipulations, the following expression for the pair correlation function $g(r, T)$:

$$g(r, T) \equiv \rho(r, T)/\rho_\infty = \tanh^2[\Lambda(r_0 - r)] \tag{29}$$

ρ_∞ being the almost temperature-independent fluid density ($=0.1450$ gm/cm^3, or 2.18×10^{-2} particles Å^3), and r_0 the temperature-independent "effective" hard-core radius ($\simeq 2.0$ Å) (Mountain and Raveché, 1973). The parameter $\Lambda = \Lambda(T)$ is given by

$$\Lambda(T) = \tilde{E}_v = \rho_\infty U \tag{30}$$

as can be inferred at once from previous manipulations.

It is important to observe that U, the strength parameter of the δ-function interaction, is temperature dependent; this is evident from equation

(30), since ρ_∞ hardly depends on the temperature. The key point here is that V is an *effective* interaction, which is a function of the properties of the medium. Clearly, with a more realistic potential, T should play a more conspicuous role.

Having obtained $g(r, T)$, we can derive the liquid structure factor $S(Q, T)$ exactly as outlined in the zero-temperature theory (Chela-Flores, 1977), taking proper care of all the analytic details (Ghassib and Khudeir, 1986). The final result is, in the small-Q limit,

$$S(Q, T) = S_0(T) + S_2(T)Q^2 \tag{31}$$

where

$$S_0(T) = 1 - 4\pi\rho_\infty r_0^3 f_1(T) \tag{32}$$

$$S_2(T) = 4\pi\rho_\infty r_0^5 f_2(T) \tag{33}$$

f_1 and f_2 being some temperature-dependent numerical factors defined by

$$\begin{aligned} f_1(T) \equiv {} & \tilde{\beta}(\tanh\alpha + 1) + \tilde{\beta}^2 I_\alpha + 2\tilde{\beta}^2 \ln 2 \\ & + \frac{\pi^2}{12}\tilde{\beta}^3 - 2\tilde{\beta}^2(\alpha \tanh\alpha - \ln\cosh\alpha) \end{aligned} \tag{34}$$

$$f_2(T) = \tilde{\beta}^3 \frac{\pi^2}{24} + \tilde{\beta}^4 \frac{3}{4}\xi(3) + \tilde{\beta}^5 \frac{7\pi^4}{1440} \tag{35}$$

In these expressions

$$\alpha \equiv \Lambda(T) r_0, \qquad \tilde{\beta} \equiv \alpha^{-1} \tag{36}$$

whereas $\xi(3)$ is the usual Riemann ζ function ($= 1.20206$), and

$$I_\alpha \equiv \int x^2 \operatorname{sech}^2 x \, dx \tag{37}$$

which is trivial to evaluate numerically.

Prior to comparing these results with the available experimental data, we should point out some of the important physical implications of the foregoing equations.

First, equation (29) still reflects the gaslike aspects of He II, long recognized by Gross (Gross, 1966) and Pitaevskii (Pitaevskii, 1961). However, this is a natural consequence of using a δ-function interaction; better results will undoubtedly arise with more realistic interactions.

Second, there should appear in the limit $T=0$ a linear term in Q. This is Feynman's term (Feynman, 1954; Feynman and Cohen, 1956), which is nothing but the phonon contribution. It is not clear whether this will appear if a realistic potential is employed, where the all-important long-range attractive tail is explicitly retained, or whether it is somehow linked with the other approximations introduced above to simplify the calculations (for example, the limit of a vanishing gauge field $\mathbf{u}$), or even if the $U(1)$ level of the effective Lagrangian $\mathcal{L}_{\text{eff}}^{(1)}$ is not adequate (cf. Section 6). Further analysis is certainly called for to resolve this question. In this connection the aim here is to do at least as well as other fairly successful attempts in the field (Feenberg, 1972).

Returning to equation (31), it is instructive to examine the low-temperature behavior ($T<0.5$ K) of $S(Q, T)$. In this regime the specific heat satisfies the familiar T^3 law, so that the entropy $\sigma\sim T^3$, as has been confirmed experimentally (Bendt et al., 1959). Moreover $\mu N\sim -T$, in the present system of units (Kittel, 1969). Thus

$$f_1\sim T, \qquad f_2\sim T \tag{38}$$

so that, in the low-T limit,

$$S(Q, T)\sim S_{0,0}+aT+bTQ^2 \tag{39}$$

where $S_{0,0}$ is just the zero-temperature result; is a nonzero constant in agreement with Price's general theory of density fluctuations (Price, 1954). The term aT is reminiscent of Goldstein's well-known thermodynamic relation (Goldstein, 1951); whereas the last term, bTQ^2, implies that the curvature of $S(Q, T)$ in the low-Q and low-T limits is linearly dependent on the temperature. This seems to conform well enough with the available experimental data (Robkoff and Hallock, 1981; Hallock, 1972; Svensson et al., 1980). Besides, except for the absence of the linear term in Q, equation (39) agrees by and large with Isihara's recent results (Isihara, 1981). It remains to be seen whether future experiments at even lower T and Q will vindicate our predictions.

5. $SU(n=2)$ GAUGE THEORY FOR HE II

5.1. Unitary Symmetry of the Boson-Pairing Lagrangian

Having gained experience with the nonzero temperature gauge theory, we see indications that perhaps one ought to explore the next step in the hierarchy of effective Lagrangians.

The next simplest approach to He II, which uses equations of evolution for macrowave functions, is a theory with a two-particle condensate, the first example of which is the theory of boson pairing of Valatin and Butler (Valatin and Butler, 1958). Subsequent studies have shown that the presence of the energy gap is a difficulty with this approach (Girardeau and Arnowitt, 1959; Wentzel, 1960; Luban, 1962).

The corresponding Lagrangian formulation of this theory may be inferred from a recent paper (Chela-Flores, 1982) (with some corrections) in analogy with the work of Nambu (Nambu, 1960; Lurie, 1970):

$$\begin{aligned}\mathscr{L}^{(2)}_{\text{eff}} &= i\phi^{\dagger}_A(\mathbf{x}, t)\,\partial_t\phi_A(\mathbf{x}, t)\\ &\quad -\frac{1}{2m_4^*}\,\partial_{\mathbf{x}}\phi^{\dagger}_A(\mathbf{x}, t)\cdot\partial_{\mathbf{x}}\phi_A(\mathbf{x}, t)-\tilde{\lambda}\phi^{\dagger}_A(\mathbf{x}, t)\phi_A(\mathbf{x}, t)\\ &\quad -\frac{U}{2}\,\phi^{\dagger}_+(\mathbf{x}, t)\phi^{\dagger}_-(\mathbf{x}, t)\phi_-(\mathbf{x}, t)\phi_+(\mathbf{x}, t)\end{aligned}\tag{40}$$

where $\mathscr{L}^{(2)}_{\text{eff}}$ denotes the effective Lagrangian of the second stage of the gauge hierarchy. For simplicity we have made the same assumption (e) as in Section 4. Here we have expressed the total field operator $\phi(\mathbf{x}, t)$ in terms of the free Hamiltonian eigenfunctions:

$$\phi_+ = \sum_{\mathbf{k}>0} a_{\mathbf{k}}(t)\,U_{\mathbf{k}}(\mathbf{x})\tag{41}$$

$$\phi_- = \sum_{\mathbf{k}>0} a_{-\mathbf{k}}(t)\,U_{-\mathbf{k}}(x)\tag{42}$$

and considered ϕ_A, $A = +$ or $-$. Just as in the case of our first step in the hierarchy (Abelian gauge field), we remark that such a Lagrangian density is invariant under global symmetry (gauge invariance of the first kind):

$$\psi_A \to \psi'_A = \exp(i\alpha)\psi_A\tag{43}$$

where α is some constant.

Bringing out the $SU(2)$ group in analogy with Nambu's (Nambu, 1960):

$$\Phi(\mathbf{x}, t) = \begin{bmatrix}\phi_+(\mathbf{x}, t)\\ \phi_-(\mathbf{x}, t)\end{bmatrix}\tag{44}$$

and with Pauli's τ_3 matrix, we may write the Lagrangian as

$$\mathscr{L}^{(2)}_{\text{eff}} = i\Phi^{\dagger}\,\partial_t\Phi - \tfrac{1}{2}\partial_{\mathbf{x}}\Phi^{\dagger}\cdot\tau_3\partial_{\mathbf{x}}\Phi - \tilde{\lambda}\Phi^{\dagger}\Phi - \tfrac{1}{2}U(\Phi^{\dagger}\tau_3\Phi)(\Phi^{\dagger}\tau_3\Phi)\tag{45}$$

5.2. Further Symmetries of the Second-Order Effective Lagrangian

We next turn the gauge invariance of the first kind (43) into a gauge invariance of the second kind, which requires a Yang–Mills gauge field $\mathbf{u}$ to compensate for the constant α acquiring space dependence.

In order to incorporate the work of Yang and Mills in $\mathscr{L}_{\text{eff}}^{(2)}$, we proceed as follows.

(a) We prefer to begin by introducing a relativistic formalism, in which space and time coordinates are treated on equal footing. We only require special relativity, unlike Anandan (1981, 1984), who coupled the Gross–Pitaevskii formalism (our zeroth approximation) to the equations of general relativity so as to study the effect of gravity on He II.

(b) Once we know how to write our $\mathscr{L}_{\text{eff}}^{(2)}$ in flat space, the nonrelativistic limit, which concerns us here, may be extracted for low-particle velocities.

(c) In flat space the $SU(2)$ gauge field may then be coupled to our matter equations in the usual way, thus justifying our long-winded way of writing $\mathscr{L}_{\text{eff}}^{(2)}$.

(d) We restrict ourselves at this level of the hierarchy to absolute zero temperature.

(e) Further, for simplicity, we discuss the Lorentz-invariant form of $\mathscr{L}_{\text{eff}}^{(0)}$. The generalization $\mathscr{L}_{\text{eff}}^{(2)}$ in (f) below is evident. Since in the Schrödinger nonrelativistic picture, $\mathbf{p}_i \to -i\partial_{x_i}$, $E \to i\partial_t$ and the nonrelativistic energy $E = p^2/2m$, we see that when using the relativistic expression $E^2 = p^2 + m^2$, we shall also be led to the Anandan-like equation (Anandan, 1981, 1984):

$$\Box\psi + m^2\psi = -U\psi|\psi|^2 \tag{46}$$

where $\Box$ denotes the d'Alembertian; however, in our case, we raise indices with Minkowski's $\eta_{\mu\nu}$, instead of Anandan's full metric $g_{\mu\nu}$.

(f) This, in turn, implies that the effective Lagrangian $\mathscr{L}_{\text{eff}}^{(0)}$ must be generalized to the relativistic form

$$\mathscr{L}_{\text{eff}}^{(0)R} = \tfrac{1}{2}\partial_\mu\psi\,\partial^\mu\psi - \tfrac{1}{2}m^2|\psi|^2 - \tfrac{1}{2}U|\psi|^4 \tag{47}$$

where for convenience we have supposed that $\tilde{\lambda} = 0$.

(g) Finally, we are led to the effective Lagrangian:

$$\begin{aligned}\mathscr{L}_{\text{eff}}^{(2)} = {} & \tfrac{1}{2}(\partial_\mu - ig\mathbf{u}_\mu\cdot\boldsymbol{\tau})\Phi^\dagger(\partial^\mu + ig\mathbf{u}^\mu\cdot\boldsymbol{\tau})\Phi \\ & -\tilde{\lambda}\Phi^\dagger\Phi - \tfrac{1}{2}U(\phi^\dagger\tau_3\Phi)(\phi^\dagger\tau_3\Phi) \\ & -\frac{k}{4}(\partial^\mu\mathbf{u}^\nu - \partial^\nu\mathbf{u}^\mu + 2g\mathbf{u}^\mu\times\mathbf{u}^\nu)^2\end{aligned} \tag{48}$$

where k is a dimensional parameter, and g is related to the structure constant of the $SU(2)$ group. For practical applications, only the limit of low-fluid velocities need be considered.

We have thus inferred a more involved gauge theory for superfluidity in which, by analogy with our experience in Sections 2 and 3, the gauge field will have an interpretation of a fluid velocity.

6. TOWARD AN $SU(n)$ THEORY

6.1. $U(1)$ Versus $SU(2)$ Gauge Theories of He II

The criticisms raised earlier (Girardeau and Arnowitt, 1959; Wentzel, 1960; Luban, 1962) to the pairing theory with no gauge fields can no longer be applied to this non-Abelian pairing theory, since we have an intrinsic (gauge) velocity field, which potentially may quench the gap, since the velocity of the liquid is the analog to the magnetic field in the case of superconductivity. We expect that applying the Madelung transformation to the field equations will give us a (non-Abelian) hydrodynamics of He II, following closely our work of Section 3. The unsatisfactory points of the present approach—namely, the low value of the condensate fraction in the zero-temperature limit (Chela-Flores, 1976) and the missing linear term in the zero-T limit of $S(Q, T)$—might then give way to a closer agreement with experiment.

However, as forcefully pointed out by March and Galasiewicz (1976), experiments cannot decide between a two- or more-particle condensate.

6.2. $SU(3)$ Gauge Theory of He II and Beyond

We have already seen in Section 5 that the earliest two-particle condensate theory, in terms of pair correlations ψ_A, $A=+, -$, is due to Valatin and Butler (Valatin and Butler, 1958). In fact, March and Galasiewicz (1976) have argued that a ground-state wave function cannot be constructed as a product of pairs if the condensate fraction vanishes. We conclude that at least three-atom correlations ought, then, to be included in the gauge hierarchy. If such were the case, then an internal $SU(3)$ space with the triplet

$$\Phi_A(x, t)=\begin{pmatrix}\phi(\mathbf{x}, t)\\ \psi(\mathbf{x}, t)\\ \xi(\mathbf{x}, t)\end{pmatrix} \tag{49}$$

should be defined, and the associated $SU(3)$ gauge theory of *bosons* would lead us, following the same steps as in this section, to the *third* stage of the hierarchy—an effective Lagrangian $\mathscr{L}_{\text{eff}}^{(3)}$. A relativistic $SU(3)$ gauge theory of fermions is well known as quantum chromodynamics (Gross and Wilczek, 1973; Politzer, 1973). However, there is no reason why bigger multiplets would not be correlated into clusters ('t Hooft, 1982), thereby leading to $SU(n)$ gauge theories. By following the method indicated in Section 5, we would be able to study the nth stage of the hierarchy, in terms of the effective Lagrangian $\mathscr{L}_{\text{eff}}^{(n)}$.

In fact, we may conjecture that as $n \to \infty$, the $SU(n \to \infty)$ gauge theory gives the correct description to superfluidity. Since n in the nonrelativistic problem of He II has a maximum value, n_{max} = the total number of helium atoms, $\sim 10^{23}$ atoms/cm^3, the mathematical limit is, in our case, a very good approximation. We have taken this limit, motivated by the conjecture ('t Hooft, 1982) that *$SU(n \to \infty)$ gauge theory is mathematically well defined.* Yet 't Hooft type of fermion gauge theories carry chiral symmetry and asymptotic freedom. None of these properties is required for an $SU(n \to \infty)$ boson gauge theory of superfluidity; it is quite plausible that, if a proof is obtained in 't Hooft's fermion case, the conjecture might be realized in our boson theory. Although it is clearly too early to decide this question by theory or experiment, we have succeeded in pointing out a scheme in which gradual progress in this field still allows us to work out properties of He II at each stage of the gauge hierarchy, as shown in Sections 3 and 4.

7. CONCLUSION

In this paper we have taken the preliminary steps toward a comprehensive theory for He II:

(i) We have generalized the theory to arbitrary temperatures.

(ii) By so doing, we have acquired a better insight into the Abelian $U(1)$ gauge field $\mathbf{u}$ itself.

(iii) We have incorporated the two-particle condensate in an $SU(2)$ gauge theory for superfluidity through the velocity field $\mathbf{u}_\mu$.

(iv) In view of difficulties of the March–Galasiewicz type, which require correlations of more than two bosons, we are led, as in Section 5, to an $SU(3)$ gauge theory, at least. The hierarchy of effective Lagrangians is conjectured to lead eventually to an $SU(n)$ gauge theory in which every particle of the superfluid participates in the n-particle correlation underlining the theory.

We feel that the new and old experimental aspects of the superfluidity of He II can find a proper description in the various stages of the gauge hierarchy of the conjectured comprehensive $SU(n \to \infty)$ gauge theory, since we have understood how the $U(1)$ simplest approximation works, and we have shown how to turn the global symmetry of the two-particle condensate theory into a local $SU(2)$ gauge symmetry of the type of Yang and Mills (Yang and Mills, 1954).

ACKNOWLEDGMENTS

The authors would like to thank Professor Abdus Salam, the International Atomic Energy Agency, and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste, where this work was started.

APPENDIX

Consider a unitary transformation that mixes the field components ψ_A (Fayet, 1982):

$$\psi_A \to U_{AB}\psi_B, \qquad A, B = 1, 2, \ldots, n \tag{A1}$$

where

$$UU^\dagger = U^\dagger U \tag{A2}$$

Here U are $(n \times n)$ unitary matrices, which may be chosen to have a determinant of 1. The set of all these matrices closes a group. It is customary to parametrize the U's by $(n^2-1)\alpha_j'$s, where

$$U_{AB} = [\exp(iT_i\alpha_i)]_{AB} \tag{A3}$$

The matrices $(T_i)_{AB}$ are elements of an algebra defined by the commutation relations

$$[T_i, T_j] = if_{ijk}T_k \tag{A4}$$

where f_{ijk} are the relevant structure constants, $ijk = 1, \ldots, n$.

The requirement that the phases α_i are coordinate dependent (i.e., local) induces a generalization of the electrodynamic gauge invariance. The requirement that the theory be invariant under a *local* $SU(n)$ [cf. equation (A3)] leads to the introduction of the analog of the vector potential, namely, an $(n \times n)$ Hermitian matrix $[u_\mu(x)]_{AB}$. There are $n^2 - 1$ guage fields $\mathbf{u}_\mu$ that form elements of the matrix u_μ:

$$[u_\mu]_{AB} = u^i_\mu(T_i)_{AB} \tag{A5}$$

The combination

$$(\partial_\mu - igu_\mu) \tag{A6}$$

is called a covariant derivative. Yang and Mills (1954) showed that the analog of the electromagnetic field $F_{\mu\nu}$ has the form

$$F^i_{\mu\nu}(x) = \partial_\mu u^i_\nu - \partial_\nu u^i_\mu + gf^i_{jk}u^j_\mu u^k_\nu \tag{A7}$$

and it transforms according to

$$F^{i}_{\mu\nu}T_i \rightarrow UF^{i}_{\mu\nu}T_iU^{\dagger} \tag{A8}$$

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